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Bianchi-I cosmology in the presence of a negative cosmological constant

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Based on

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A generalized Heckmann-Schucking cosmological solution in the presence of a negative cosmological constant

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Content

Motivation

There are two most important classes of cosmological solutions.

The Friedmann-Robertson-Walker isotropic cosmological models, which constitute a basis for comparison of theoretical predictions with observations .

The anisotropic Kasner solution for the empty Bianchi-I universe. Its importance is connected with its role in the description of the oscillatory approach to the cosmological singularity BKL.

The Heckmann - Schucking anisotropic solution for the Bianchi-I universe in the presence of the dust-like matter constitutes a bridge between these two types of the cosmological solutions: in the vicinity of the cosmological singularity it behaves as a Kasner universe, while at the later stage of the cosmological evolution it behaves as an isotropic flat Friedmann Universe.

The Heckmann-Schucking solution has the following form: for the Bianchi-I universe with the metric

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2$$

filled with dust whose equation of state is

$$p = 0$$

$$a(t) = a_0 t^{p_1}(t + t_0)^{2/3 - p_1},$$
$$b(t) = b_0 t^{p_2}(t + t_0)^{2/3 - p_2},$$
$$c(t) = c_0 t^{p_3}(t + t_0)^{2/3 - p_3},$$

The exponents p_1, p_2 and p_3 are the Kasner exponents:

$$p_1 + p_2 + p_3 = 1,$$
$$p_1^2 + p_2^2 + p_3^2 = 1.$$

Usually the Kasner exponents are arranged in such a way that

$$p_1 \leq p_2 \leq p_3$$

In our paper (I.M. Khalatnikov and A.Yu. Kamenshchik, 2003) the Heckmann-Schucking solution was generalized on the case of the Bianchi-I universe filled with a mixture of three perfect fluids: dust, stiff matter with the equation of state $p = \rho$, dust and a positive cosmological constant.

Why a positive cosmological constant is of interest ?

The recent discovery of the phenomenon of the cosmic acceleration.

Inflationary cosmology.

Why a negative cosmological constant might be of interest ?

It is compatible with the supersymmetry.

It could be reconciled with modern observations.

It implies a cosmological scenario, where the present expansion of the universe is followed by a contraction, ending in a Big Crunch cosmological singularity. Thus, one has a universe, existing during a finite period of cosmic time between two cosmological singularities.

Model

The Bianchi-I universe filled with three fluids: dust, stiff matter and a negative cosmological constant with the equation of motion:

$$p = -\rho, \quad \rho = -\Lambda, \quad \Lambda > 0.$$

The construction of the solution

$$a(t) = R(t)\exp(-2\alpha(t)),$$
$$b(t) = R(t)\exp(\alpha(t) - \beta(t)),$$
$$c(t) = R(t)\exp(\alpha(t) + \beta(t)),$$

$R(t)$ is the conformal factor,
$\alpha(t)$ and $\beta(t)$ characterize the anisotropy of the model.

$$R_0^0 = -\left(3\frac{\ddot{R}}{R} + 6\dot{\alpha}^2 + 2\dot{\beta}^2\right),$$

$$R_1^1 = -\left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} - 6\frac{\dot{R}}{R}\dot{\alpha} - 2\ddot{\alpha}\right),$$

$$R_2^2 = -\left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + 3\frac{\dot{R}}{R}(\dot{\alpha} - \dot{\beta}) + \ddot{\alpha} - \ddot{\beta}\right),$$

$$R_3^3 = -\left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + 3\frac{\dot{R}}{R}(\dot{\alpha} + \dot{\beta}) + \ddot{\alpha} + \ddot{\beta}\right).$$

Using the isotropy of the energy-momentum tensor one has

$$R_1^1 = R_2^2 = R_3^3.$$

$$\ddot{\alpha} + 3\frac{\dot{R}}{R}\dot{\alpha} = 0,$$

$$\ddot{\beta} + 3\frac{\dot{R}}{R}\dot{\beta} = 0.$$

$$\dot{\alpha} = \frac{\alpha_0}{R^3},$$

$$\dot{\beta} = \frac{\beta_0}{R^3}.$$

The 00 component of the Einstein equations has now the form

$$\frac{\dot{R}^2}{R^2} = \dot{\alpha}^2 + \frac{\dot{\beta}^2}{3} + \frac{4\pi G}{3}\rho.$$

The effective Friedmann equation

$$\frac{\dot{R}^2}{R^2} = \dot{\alpha}^2 + \frac{\dot{\beta}^2}{3} - \Lambda + \frac{M}{R^3} + \frac{S}{R^6},$$

$$\frac{\dot{R}^2}{R^2} = -\Lambda + \frac{M}{R^3} + \frac{S_0}{R^6},$$

$$S_0 = S + \alpha_0^2 + \frac{\beta_0^2}{3}.$$

$$R^3(t) = \frac{M}{2\Lambda} + \sqrt{\frac{S_0}{\Lambda}}\sin(3\sqrt{\Lambda}t) - \frac{M}{2\Lambda}\cos(3\sqrt{\Lambda}t).$$

$$R^3(0) = 0.$$

$$R^3(t) = \frac{M}{2\Lambda} + \frac{\sqrt{4S_0\Lambda + M^2}}{2\Lambda}\sin\left[3\sqrt{\Lambda}t - \arcsin\left(\frac{M}{\sqrt{4S_0\Lambda + M^2}}\right)\right].$$

The Big Crunch singularity is encountered at

$$t_{BC} = \frac{1}{3\sqrt{\Lambda}}\left[\pi + 2\arcsin\left(\frac{M}{\sqrt{4S_0\Lambda + M^2}}\right)\right].$$

$$\alpha(t) = \frac{\alpha_0}{3\sqrt{S_0}} \ln \left[\frac{\frac{2\sqrt{S_0}}{3\Lambda t_\alpha} \sin \frac{3\sqrt{\Lambda} t}{2}}{\frac{M}{2\Lambda} \sin \frac{3\sqrt{\Lambda} t}{2} + \sqrt{\frac{S_0}{\Lambda}} \cos \frac{3\sqrt{\Lambda} t}{2}} \right].$$

The integration constant is chosen in such a way to provide a Kasner-type behaviour of the function $\alpha(t)$ in the neighbourhood of the Big Bang singularity.

$$\lim_{t \to 0} \alpha(t) = \frac{\alpha_0}{3\sqrt{S_0}} \ln t.$$

$$\beta(t) = \frac{\beta_0}{3\sqrt{S_0}} \ln \left[\frac{\frac{2\sqrt{S_0}}{3\Lambda t_\beta} \sin \frac{3\sqrt{\Lambda} t}{2}}{\frac{M}{2\Lambda} \sin \frac{3\sqrt{\Lambda} t}{2} + \sqrt{\frac{S_0}{\Lambda}} \cos \frac{3\sqrt{\Lambda} t}{2}} \right].$$

$$a(t)=\left(\frac{\sqrt{S_0}}{3\Lambda t_\alpha}\right)^{-\frac{2\alpha_0}{3\sqrt{S_0}}}\cdot\left(2\sin\frac{3\sqrt{\Lambda}t}{2}\right)^{\left(\frac{1}{3}-\frac{2\alpha_0}{3\sqrt{S_0}}\right)}$$

$$\cdot\left(\frac{M}{2\Lambda}\sin\frac{3\sqrt{\Lambda}t}{2}+\sqrt{\frac{S_0}{\Lambda}}\cos\frac{3\sqrt{\Lambda}t}{2}\right)^{\left(\frac{1}{3}+\frac{2\alpha_0}{3\sqrt{S_0}}\right)},$$

$$b(t)=\left(\frac{\sqrt{S_0}}{3\Lambda t_\alpha}\right)^{\frac{\alpha_0}{3\sqrt{S_0}}}\left(\frac{\sqrt{S_0}}{3\Lambda t_\beta}\right)^{-\frac{\beta_0}{3\sqrt{S_0}}}\cdot\left(2\sin\frac{3\sqrt{\Lambda}t}{2}\right)^{\left(\frac{1}{3}+\frac{\alpha_0-\beta_0}{3\sqrt{S_0}}\right)}$$

$$\cdot\left(\frac{M}{2\Lambda}\sin\frac{3\sqrt{\Lambda}t}{2}+\sqrt{\frac{S_0}{\Lambda}}\cos\frac{3\sqrt{\Lambda}t}{2}\right)^{\left(\frac{1}{3}-\frac{\alpha_0-\beta_0}{3\sqrt{S_0}}\right)},$$

$$
\begin{aligned}
c(t) = {} & \left(\frac{\sqrt{S_0}}{3\Lambda t_\alpha}\right)^{\frac{\alpha_0}{3\sqrt{S_0}}} \left(\frac{\sqrt{S_0}}{3\Lambda t_\beta}\right)^{\frac{\beta_0}{3\sqrt{S_0}}} \cdot \left(2\sin\frac{3\sqrt{\Lambda}t}{2}\right)^{\left(\frac{1}{3}+\frac{\alpha_0+\beta_0}{3\sqrt{S_0}}\right)} \\
& \cdot \left(\frac{M}{2\Lambda}\sin\frac{3\sqrt{\Lambda}t}{2} + \sqrt{\frac{S_0}{\Lambda}}\cos\frac{3\sqrt{\Lambda}t}{2}\right)^{\left(\frac{1}{3}-\frac{\alpha_0+\beta_0}{3\sqrt{S_0}}\right)} .
\end{aligned}
$$

At small values of t:

$$a(t) \sim t^{p_1},$$
$$b(t) \sim t^{p_2},$$
$$c(t) \sim t^{p_3},$$

$$p_1 = \frac{1}{3} - \frac{2\alpha_0}{3\sqrt{S_0}},$$
$$p_2 = \frac{1}{3} + \frac{\alpha_0 - \beta_0}{3\sqrt{S_0}},$$
$$p_3 = \frac{1}{3} + \frac{\alpha_0 + \beta_0}{3\sqrt{S_0}}.$$

$$p_1 + p_2 + p_3 = 1.$$

$$p_1^2 + p_2^2 + p_3^2 = 1 - q^2,$$
$$q^2 = \frac{2S}{3S_0} = \frac{2S}{3\left(S + \alpha_0^2 + \frac{\beta_0^2}{3}\right)},$$

where
$0 \leq q^2 \leq \frac{2}{3}$.
This relation was obtained in the paper by
V.A. Belinsky and I.M. Khalatnikov, 1973.

The Lifshitz-Khalatnikov parameter

If $q^2 = 0$ then

$$p_1 = -\frac{u}{1+u+u^2},$$
$$p_2 = \frac{1+u}{1+u+u^2},$$
$$p_3 = \frac{u(1+u)}{1+u+u^2},$$

where $u \geq 1$.

In the vicinity of the Big Crunch singularity,

$$t \longrightarrow t_{BC}$$

$$a(t) \sim (t_{BC} - t)^{\left(\frac{2}{3} - p_1\right)},$$
$$b(t) \sim (t_{BC} - t)^{\left(\frac{2}{3} - p_2\right)},$$
$$c(t) \sim (t_{BC} - t)^{\left(\frac{2}{3} - p_3\right)}.$$

The axes, corresponding to the Kasner exponents p_1 and p_3 exchange their roles.

$$a(t) \sim (t_{BC} - t)^{p_3'},$$
$$b(t) \sim (t_{BC} - t)^{p_2'},$$
$$c(t) \sim (t_{BC} - t)^{p_1'},$$
$$p_1' \leq p_2' \leq p_3'.$$

The transformation of the Lifshitz-Khalatnikov parameter

$$p_1' = -\frac{u'}{1+u'+u'^2},\ \ p_2' = \frac{1+u'}{1+u'+u'^2},$$
$$p_3' = \frac{u'(1+u')}{1+u'+u'^2}.$$

$$u' = \frac{u+2}{u-1}.$$

For comparison

In the oscillatory approach to the singularity the change of a Kasner epoch, i.e. the change of roles of scale functions, corresponding to the Kasner exponents p_1 and p_2 is combined with the shift

$$u' = u - 1.$$

The change of a Kasner era, when the axes, corresponding to the exponents p_2 and p_3 exchange their roles is combined with the transformation of the parameter u is transformed into

$$u' = \frac{1}{u}.$$

The properties of the solution

The function $a(t)$ begin its evolution contracting and finish it also in the contraction phase. Thus, it can have or two extrema (minimum and maximum values) or none.

The function $c(t)$ increases both at the beginning and at the end of the cosmological evolution, and, hence also it has two or none of the extrema.

The function $b(t)$ increases in the vicinity of the initial singularity and decreases in the vicinity of the final singularity. It has one maximum value.

The estremum condition

$$\sin\left(3\sqrt{\Lambda}t_{i\ ext} + \arcsin\frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2+4\Lambda S_0}}\right) = (1-3p_i)\frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2+4\Lambda S_0}}$$
$i = 1, 2, 3.$

For the function $b(t)$:

$$t_{2\ max} = \frac{1}{3\sqrt{\Lambda}}\left(\pi + \arcsin\left[(3p_2-1)\frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2+4\Lambda S_0}}\right]\right.$$
$$\left.-\arcsin\frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2+4\Lambda S_0}}\right).$$

For the function $a(t)$ the extremum solutions exist if

$$(1-3p_1)\frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2+4\Lambda S_0}} \leq 1.$$

If

$$M^2 \geq 12\Lambda S_0.$$

these solutions always exist.

$$t_{1\ min} = \frac{1}{3\sqrt{\Lambda}} \left(\arcsin \left[(1 - 3p_1) \frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2 + 4\Lambda S_0}} \right] \right.$$
$$\left. - \arcsin \frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2 + 4\Lambda S_0}} \right),$$
$$t_{1\ max} = \frac{1}{3\sqrt{\Lambda}} \left(\pi - \arcsin \left[(1 - 3p_1) \frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2 + 4\Lambda S_0}} \right] \right.$$
$$\left. - \arcsin \frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2 + 4\Lambda S_0}} \right).$$

Similarly for $c(t)$:

$$t_{3\ max} = \frac{1}{3\sqrt{\Lambda}}\left(\pi + \arcsin\left[(3p_3 - 1)\frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2 + 4\Lambda S_0}}\right]\right.$$
$$\left. -\arcsin\frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2 + 4\Lambda S_0}}\right),$$
$$t_{3\ min} = \frac{1}{3\sqrt{\Lambda}}\left(2\pi - \arcsin\left[(3p_2 - 1)\frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2 + 4\Lambda S_0}}\right]\right.$$
$$\left. -\arcsin\frac{\sqrt{4\Lambda S_0}}{\sqrt{M^2 + 4\Lambda S_0}}\right).$$

Conclusion

The generalized Heckmann-Schucking solution in the presence of a negative cosmological term can be interpreted as a very simplified model of a Bianchi-IX universe, having chaotic oscillatory regimes at the beginning and at the end of its evolution. It would be very interesting to find the relations connecting characteristics of these two regimes for a Bianchi-IX universe.

Afterword: singularities and observations Tachyons and the Big Brake singularity

Based on
Z. Keresztes, L.A. Gergely, V. Gorini, U. Moschella and A.Yu. Kamenshchik,
Tachyon cosmology, supernovae data and the Big Brake singularity,
Phys. Rev. D79 (2009) 083504

Tachyons and the Big Brake singularity

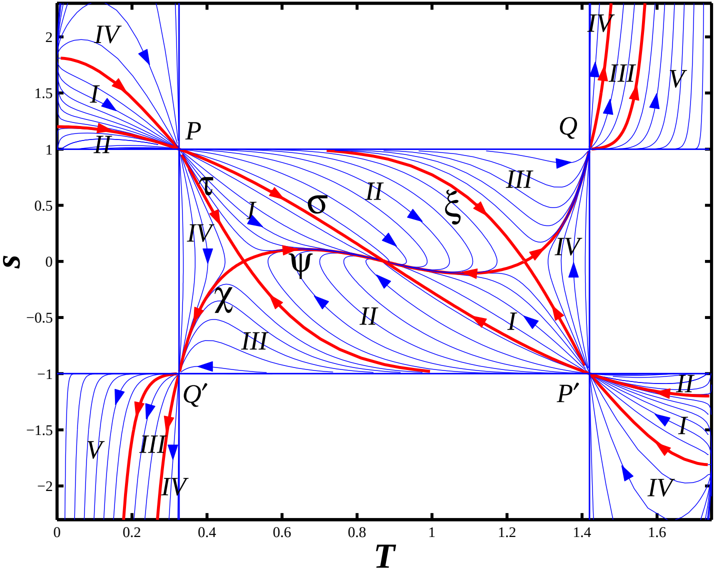
$$L = -V(T)\sqrt{1-\dot{T}^2},$$

$$\varepsilon = \frac{V(T)}{\sqrt{1-\dot{T}^2}}$$

$$p = -V(T)\sqrt{1-\dot{T}^2}.$$

$$V(T) = \frac{\Lambda}{\sin^2\left(\frac{3}{2}\sqrt{\Lambda(1+k)}T\right)}$$

$$\times\sqrt{1-(1+k)\cos^2\left(\frac{3}{2}\sqrt{\Lambda(1+k)}T\right)}.$$



Phase portrait evolution for $k > 0, s \equiv \dot{T}$

Big Brake cosmological singularity

$$ds^2 = dt^2 - a^2(t)dl^2$$

$$t \to t_{BB} < \infty$$

$$a(t \to t_{BB}) \to a_{BB} < \infty$$

$$\dot{a}(t \to t_{BB}) \to 0$$

$$\ddot{a}(t \to t_{BB}) \to -\infty$$

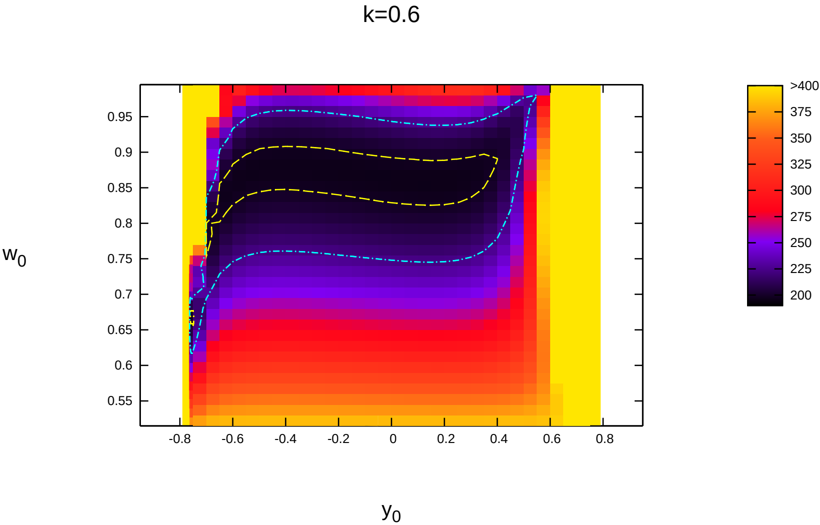
$$R(t \to t_{BB}) \to +\infty$$

$$T(t \to t_{BB}) \to T_{BB},\ |T_{BB}| <$$

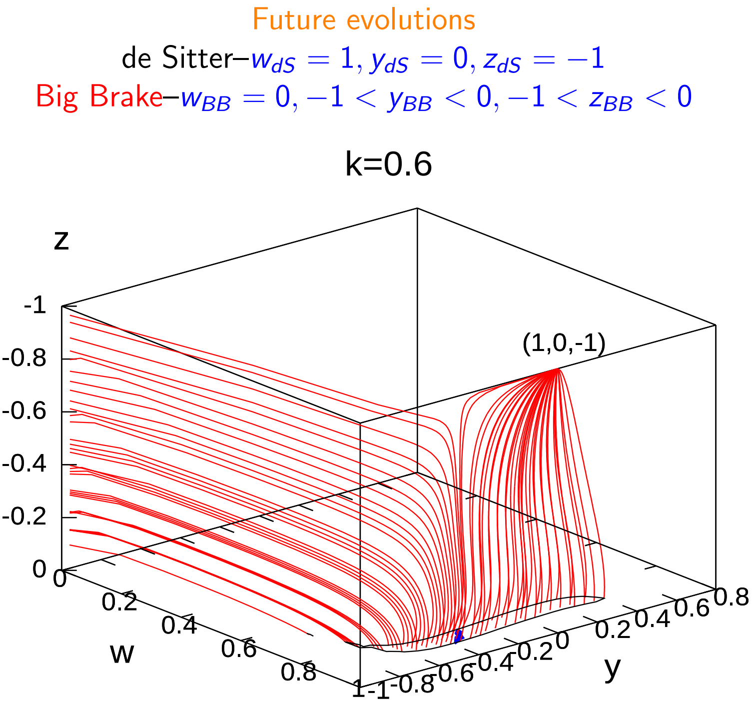
$$|s(t \to t_{BB})| \to \infty$$

Is the Big Brake evolution in our model compatible with supernovae type Ia data ?

1. We select the compatible initial conditions by studying the backward evolution in comparison with the luminosity - redshift diagrams for the supernovae type Ia standard(izable) candles.
2. Choosing initial conditions which are compatible at the 1σ level with the data, we study the forward evolution and show that a deceleration period following the present accelerated expansion is possible, and when it is so, we estimate how long it is expected to last.



The fit of the luminosity distance vs. redshift in the parameter plane ($y_0 = \cos\left(\frac{3}{2}\sqrt{\Omega_\Lambda(1+k)}H_0\,T\right)$, $w_0 = 1/\left(1+s_0^2\right)$). The white areas represent regions where the bounds on the model are not satisfied. The contours refer to the 68.3% (1σ) and 95.4% (2σ) confidence levels. The colour code for χ^2 is indicated on the vertical stripes.


Future evolutions
de Sitter–$w_{dS} = 1, y_{dS} = 0, z_{dS} = -1$
Big Brake–$w_{BB} = 0, -1 < y_{BB} < 0, -1 < z_{BB} < 0$
k=0.6
z
-1
-0.8
-0.6
-0.4
-0.2
0
(1,0,-1)
w
0
0.2
0.4
0.6
0.8
1
y
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

When the Big Brake will come ?

y_0	w_0	z_*	$t_*\,(10^9 yrs)$	z_{BB}	$t_{BB}\,(10^9 yrs)$
-0.70	0.770	-0.100	1.5	-0.145	2.3
-0.65	0.815	-0.168	2.6	-0.209	3.4
-0.60	0.830	-0.240	3.9	-0.277	4.7
-0.60	0.875	-0.261	4.2	-0.296	4.0
-0.55	0.875	-0.347	5.9	-0.377	6.7
-0.50	0.860	-0.427	7.8	-0.453	8.6
-0.45	0.860	-0.533	10	-0.554	11
-0.45	0.905	-0.616	13	-0.633	14
-0.40	0.890	-0.733	18	-0.745	19
-0.35	0.860	-0.814	23	-0.822	24
-0.35	0.875	-0.865	28	-0.872	29
-0.35	0.890	-0.927	36	-0.930	37
-0.30	0.845	-0.955	43	-0.957	44